JOURNAL OF ALGEBRA 76, 186–191 (1982)

# Generation and Projective Generation of Ideals

MICHAEL RANDY GABEL

*Department of Mathematics, George Mason University, Fairfax, Virginia 22030*

*Communicated by D. Buchsbaum*

Received November 4, 1980

In this paper, rings are commutative with identity. They need not be Noetherian, except as noted. The minimal number of generators of a finitely generated $R$-module $M$ is denoted by $v(M)$.

We say that $M$ has $n$ generators if $v(M) \leqslant n$ This is clearly equivalent to the existence of an epimorphism $R^n \to M$. By analogy we say that $M$ has $n$ projective generators if there is an epimorphism $P \to M$ with $P$ a projective $R$-module of rank $n$.

The papers by Swan [12] and Serre [10] give early examples of generation and projective generation theorems. New results on the structure of projective modules have prompted a rash of projective generation theorems, some of which have become generation theorems as result of the projective in question being shown to be free. See, for example, papers by Boratynski [2, 3], Mohan Kumar [8], Geramita-Weibel [5], and the lecture notes by Vasconcelos [14].

In this paper we prove a projective generation and a generation theorem (Theorem 2) which are essentially extensions of Boratynski's projective generation theorem [2]. We deduce as a corollary a theorem due to Mohan Kumar (Corollary 3). The techniques are sufficiently different from those of Mohan Kumar (stable range versus prime avoidance) that improvements may be possible.

The proofs of these results prompt some questions on the map $K_1(R_{fg}) \to K_0(R)$ (which is defined when $(f, g) = R$), and on conditions on $g$ for which $P_g$ is free. The paper concludes with some results on these questions.

We begin with a proposition due to Boratynski. We omit the proof.

PROPOSITION 1 (Boratynski [2]). *Let $R$ be a ring and let $I \subset R$ be an ideal. Set $n = v(I/I^2)$.*

*Then there exist $x_1, \ldots, x_n \in I$ and $s, s' \in R$ such that*

(1) $(s, s') = R$,

(2) $ss' \in (x_1, \ldots, x_n)$,



0021-8693/82/050186–06\$02.00/0

(3) $I_s = (x_1, \ldots, x_n) R_s$,

(4) $I_{s'} = R_{s'}$.

The following theorem is the extension of Boratynski's projective generation theorem.

THEOREM 2. *Let $R$, $I$, $x_1, \ldots, x_n$, $s$, $s'$ be as in the above proposition. Set $P = \ker(R^n_{ss'} \to^{(x_1 \cdots x_n)} R_{ss'})$. Then*

(1) *If $P$ is isomorphic to the localization at $s$ of a projective $R_{s'}$-module, then $I$ has $n$ projective generators.*

(2) *If there exists an element $g \in R$ such that $(ss', g) = R$ and $x_1, \ldots, x_n$ becomes stable in $R_{ss'g}$, then $I$ has $n$ generators, that is, $v(I) = v(I/I^2)$.*

*Proof.* Boratynski proves something weaker than (1) in [2], where he proves that $I$ has $n$ projective generators if there is a projective $R_{s'}$-module $Q$ which localizes to $P$ at $s$ which satisfies the additional condition that $Q \oplus R_{s'} \approx R^n_{s'}$.

To prove (1), we have by assumption a projective $R_{s'}$-module, $Q$, for which there is an isomorphism $f: P \to Q_s$. We will show that there is an isomorphism $h: R^n_{ss'} \to (Q \oplus R_{s'})_s$ such that the projective module over $R$ obtained by patching $R^n_s$ and $Q \oplus R_{s'}$ via $h$ over $R_{ss'}$ maps onto $I$.

To do this, consider the maps $R^n_s \to^{(x_1 \cdots x_n)} (x_1, \ldots, x_n) R_s = I_s$ and $Q \oplus R_{s'} \to^{\pi_2} R_{s'} = I_{s'}$. Choose a splitting $\sigma: R^n_{ss'} \to P$ of the inclusion $P \subset R^n_{ss'}$. Define $h$ to be the composite of the isomorphisms $R^n_{ss'} \to P \oplus R_{ss'} \to Q_s \oplus R_{ss'}$ given by $h(\varepsilon_i) = (f(\sigma(\varepsilon_i)), x_i)$.

Since the following diagram commutes, we are done with part (1).

$$\begin{tikzcd}
R^n_{ss'} \arrow[rr, "(x_1 \cdots x_n)"] \arrow[d, "(\sigma, (x_1 \cdots x_n))"] & & (x_1 \cdots x_n) R_{ss'} = I_{ss'} \arrow[dd, equal] \\
h \qquad P \oplus R_{ss'} \arrow[d, "f \oplus 1"] & & \\
Q_s \oplus R_{ss'} \arrow[rr, "\pi_2"] & & R_{ss'} = I_{ss'}
\end{tikzcd}$$

To prove (2), first note that $(s, s') = R$ and $(ss', g) = R$ implies $(s, s'g) = R$. Consider now the maps

$$R^n_s \xrightarrow{(x_1 \cdots x_n)} (x_1, \ldots, x_n) R_s = I_s$$

and

$$R^n_{s'g} \xrightarrow{(10 \cdots 0)} R_{s'g} = (R_{s'})_g = (I_{s'})_g = I_{s'g}.$$

Since $x_1,\ldots,x_n$ is stable in $R_{ss'g}$ (i.e., since there exist elements $r_1,\ldots,r_{n-1}$ in $R_{ss'g}$ such that the sequence $x_1+r_1x_n,\ldots,x_{n-1}+r_{n-1}x_n$ is unimodular), there is an *elementary* matrix $E \in GL(n, R_{ss'g})$ making the following diagram commute:

$$\begin{tikzcd} R^n_{ss'g} \arrow[r, "(x_1\cdots x_n)"] \arrow[d, "E"'] & I_{ss'g} \arrow[d, equal] = R_{ss'g} \\ R^n_{ss'g} \arrow[r, "(10\cdots 0)"] & I_{ss'g} = R_{ss'g} \end{tikzcd}$$

Therefore $I$ is the image of a projective $R$-module obtained by patching two free modules via an elementary matrix.. But by a theorem of Suslin [11] (see Proposition 5, also), such projectives are free and thus $I$ has $n$ generators. Q.E.D.

*Remarks.* Note that the hypothesis in part (1) of the theorem is satisfied if $P$ is free, i.e., if $x_1,\ldots,x_n$ is completable over $R_{ss'}$. It is also satisfied if there is an element $g \in R$ such that $(ss', g) = R$ and $P_g$ is free (not necessarily because of stability) over $R_{ss'g}$. For in this case $P$ extends to a projective module over *all* of $R$: just patch $P$ and $R_g^{n-1}$ over $R_{ss'g}$ via the assumed isomorphism between $P_g$ and $R^{n-1}_{ss'g}$.

We now deduce as a corollary a theorem due to Mohan Kumar [8]. As the techniques are based on stable range rather than dimension, refinements in his theorem may be possible.

COROLLARY 3 (Mohan Kumar). *Let $R$ be a Noetherian ring and $I \subset R$ an ideal satisfying $v(I/I^2) > \dim R$. Then $v(I) = v(I/I^2)$.*

*Proof.* This follows immediately from Theorem 2, part (2) and the following lemma whose proof has been gleaned from an argument of Murthy's ([9, Theorem 3.1])

LEMMA 4. *Let $R$ be a Noetherian ring and $f \in R$. Say $x_1,\ldots,x_n \in R_f$ is a unimodular sequence. Assume $n > \dim R$. Then there is a $g \in R$ such that $(f, g) = R$ and $x_1,\ldots,x_n$ is stable in $R_{fg}$.*

*Proof.* Consider the multiplicative set $S = 1 + (f)$. Since every maximal ideal of $R$ either contains $f$ or an element of the form $1 + xf$, $\dim S^{-1}R_f < \dim R$. So Bass's stable range theorem [1] applies to $x_1,\ldots,x_n$ in $S^{-1}R_f$ as $n-1 > \dim S^{-1}R_f$. So $x_1,\ldots,x_n$ is stable in $S^{-1}R_f$. But then there is a $g \in S$ such that $x_1,\ldots,x_n$ is stable in $R_{fg}$. Since $g \in 1 + (f)$, we are done. Q.E.D.

*Remark.* Suslin's proof of the fact that when you patch two free modules via an elementary matrix you get a free module is rather technical so we give here a proof of the weaker proposition that the resulting module is stably free. This result is sufficient to deduce Mohan Kumar's result as, although the patched module produced in the proof of Theorem 2, part (2), would now be known to be only stable free, under the hypothesis $v(I/I^2) > \dim R$ it would have rank sufficiently high (namely, $n$) to be free by the Bass Cancellation Theorem [1]. Also the proof we give avoids the assumption of Gersten [6] that $R$ be "$K$-regular," i.e., that for any set $X$, $K_0(R) \to K_0(R[X])$ be an isomorphism. That such a patched module, as above, be stably free is equivalent to the statement that the map $d: GL(R_{fg}) \to K_0(R)$ given y $d(h) = [P(h)] - [R^n]$ (where $h \in GL(n, R_{fg})$ and $P(h)$ is the projective $R$-module gotten by patching $R_f^n$ and $R_g^n$ over $R_{fg}$ via $h$) factors through $K_1(R_{fg}) \equiv GL(R_{fg})/E(R_{fg})$. But since $E(R_{fg}) = [GL(R_{fg}), GL(R_{fg})]$ and $K_0(R)$ is Abelian, this will follow if $d$ is shown to be a homomorphism. This is done in the following.

Proposition 5. *Let $R$ be a ring and let $f, g \in R$ with $(f, g) = R$. Let $\alpha, \beta \in GL(n, R_{fg})$. Then $P(\alpha\beta) \oplus R^n \approx P(\alpha) \oplus P(\beta)$.*

*Proof.* As in Gersten [6], for example, it suffices to find $x \in GL(2n, R_g)$ and $y \in GL(2n, R_f)$ such that $x\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)y = \left(\begin{smallmatrix} \alpha\beta & 0 \\ 0 & I \end{smallmatrix}\right)$, where $I$ is the $n \times n$ identity matrix. To this end choose an integer $s$ so large that $\alpha = A/(fg)^s$, $\beta = B/(fg)^s$ with $A$ and $B$ having entries in $R$ and that (since $\alpha, \beta \in GL(n, R_{fg})$ there exist $n \times n$ matrices over $R$, $\bar{A}$ and $\bar{B}$, with $A/(fg)^s \cdot \bar{A}/(fg)^s = I$ and $B/(fg)^s \cdot \bar{B}/(fg)^s = I$. Since $(f, g) = R$, $(f^{2s}, g^{2s}) = R$. So choose $u$ and $v$ in $R$ with $uf^{2s} + vg^{2s} = 1$. Set

$$x = \begin{bmatrix} g^s I & \dfrac{u}{g^s} A \\ -\dfrac{1}{g^s}\bar{A} & vg^s I \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} \dfrac{v}{f^s} B & -uf^s I \\ f^s I & \dfrac{1}{f^s}\bar{B} \end{bmatrix}.$$

It is easy to check that $x$ and $y$ are as desired. Q.E.D.

*Remarks.* (1) Note that $P(\alpha\beta) \oplus R^n \approx P(\alpha) \oplus P(\beta)$ is stronger than the statement $d(\alpha\beta) = d(\alpha) + d(\beta)$ for the latter just says that $P(\alpha\beta) \oplus R^n$ and $P(\alpha) \oplus P(\beta)$ are *stably* isomorphic.

(2) This proposition together with Suslin's result shows that $P(\alpha E) \oplus R^n \approx P(\alpha) \oplus R^n$ if $\alpha$, $E \in GL(n, R_{fg})$ and $E$ is elementary. I do not know if the $R^n$'s can be left off.

(3) It is easy to show that if $E$ is an elementary *transvection*, then $P(E)$ is free. See [6].

(4) The $K$-theory exact sequence for the diagram

$$\begin{array}{ccc} R & \rightarrow & R_f \\ \downarrow & & \downarrow \\ R_f & \rightarrow & R_{fg} \end{array}$$

is also discussed in Landsburg [7].

Part (1) of Theorem 2 prompts questions on the "extension" problem. We offer the following proposition which is the stable version of the argument of Murthy's in [9]. The stable version does not require regularty.

PROPOSITION 6. *Let $R$ be a Noetherian ring with* $\dim R < \infty$. *Let $f \in R$ and let $P$ be a stably free $R_f$-module of* rank $= \dim R$.

*Then there is a projective $R$-module $Q$ such that $Q \oplus R$ is free and $Q_f \approx P$.*

*Proof.* Note that if rank $P > \dim R$, then $P$ is free, so there is no problem and if rank $P < \dim R$, the proposition is false (even for $R$ a polynomial ring over a field) by an example of Eisenbud's [9].

Set $\dim R = n - 1$. Now $\operatorname{rank}(P \oplus R_f) = \dim R + 1 > \dim R_f$. Since $P$ is stably free, $P \oplus R_f \approx R_f^n$. So there is a unimodular sequence $x_1, ..., x_n \in R_f$ such that $P = \ker(R_f^n \to^{(x_1 \cdots x_n)} R_f)$. By Lemma 4, there is a $g \in R$ with $(f, g) = R$ and such that $x_1, ..., x_n$ is stable over $R_{fg}$. As in the proof of Theorem 2 part (2), there is an elementary matrix $E \in E(R_{fg})$ such that $(x_1 \cdots x_n) = (10 \cdots 0) E$. Therefore, there is an isomorphism $h: P_g \to R_{fg}^{n-1}$ making the following diagram commute.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_g & \longrightarrow & R_{fg}^n & \xrightarrow{(x_1 \cdots x_n)} & R_{fg} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle E} & & \downarrow & & \\ 0 & \longrightarrow & R_{fg}^{n-1} & \longrightarrow & R_{fg}^n & \xrightarrow{(10 \cdots 0)} & R_{fg} & \longrightarrow & 0 \end{array}$$

Now patch $P$ and $R_g^{n-1}$ via $h$ over $R_{fg}$ and call the resulting module $Q$. By construction $Q$ extends $P$. Denote by $F$ the projective $R$-module obtained by patching $R_f^n$ and $R_g^n$ over $R_{fg}$ via $E$. The above commutative diagram shows that $Q \oplus R \approx F$. As before, since $E$ is elementary, $F$ is free. Q.E.D.

We conclude with some comments about a question alluded to following the proof of Theorem 2: if $P$ is projective, when is $P_g$ free? The following lemma is easy to prove.

LEMMA 7. *If $M$ is a finitely generated $R$-module, then $M_g$ has a free summand over $R_g$ if and only if there exists an ideal $I \subset R$ with $g^n \in I$ for some $n$ and an epimorphism $M \to I$.*

COROLLARY 8. *If $P \oplus R \approx R^3$, $P_g$-free if and only if there is an epimorphism $P \to I$ with $I \subset R$ an ideal satisfying $g^n \in I$ for some $n$.*

*Proof.* In view of the previous lemma, the point is that stably free projectives of rank one are free.

COROLLARY 9. *Let $R = \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$, where $\mathbb{R}$ = the reals. Let $P = \ker(R^3 \to^{(x,y,z)} R)$. Let $g \in R$. Then $P_g$ is free if and only if $g$ has a (real) zero on $S^2$.*

*Proof.* First assume $g$ has a zero on $S^2$. By arguments similar to those given in [4, Lemma 4.2] we can assume that $g(0, 0, 1) = 0$. Set $\mathbf{m} = (x, y, z - 1)$. That same paper proves that there is an epimorphism $P \to I$ to an ideal of $R$ satisfying $\mathbf{m}^3 \subset I \subset \mathbf{m}$. (This is the second example following Corollary 5.3.) But $g(0, 0, 1) = 0$, so $g \in \mathbf{m}$. Hence $g^3 \in \mathbf{m}^3 \subset I$. But then Lemma 7 shows that $P_g$ is free.

Conversely, if $g$ has no zero on $S^2$, then the ring $R_g$ maps to the ring of real-valued continuous functions on $S^2$, $C(S^2)$. Since $P_g \otimes_{R_g} C(S^2)$ is not free [13], $P_g$ is not free. Q.E.D.

## REFERENCES


1. H. BASS, "Algebraic $K$-theory," Benjamin, New York, 1968.
2. M. BORATYNSKI, A note on the set-theoretical complete intersection ideals, *J. Algebra* **54** (1978), 1–5.
3. M. BORATYNSKI, When is an ideal generated by a regular sequence?, *J. Algebra* **57** (1979), 236–241.
4. M. R. GABEL, Algebraic vector fields on $S^2$, *J. Pure Appl. Algebra* **11** (1977), 29–40.
5. A. V. GERAMITA AND C. A. WEIBEL, Ideals with trivial conormal bundle, *Canad. J. Math.* **32**, No. 1 (1980), 210–218.
6. S. M. GERSTEN, A Mayer–Vietoris sequence in the $K$-theory of localizations, *J. Pure Appl. Algebra* **2** (1972), 275–785.
7. S. E. LANDSBURG, Patching theorems for projective modules, preprint.
8. N. MOHAN KUMAR, On two conjectures about polynomial rings, *Invent. Math.* **46** (1978), 225–236.
9. M. P. MURTHY, Projective modules over certain regular polynomial rings, unpublished.
10. J. P. SERRE, Sur les modules projectifs, Sem. Dubreil–Pisot (1960/61), No. 2.
11. A. A. SUSLIN, On the structure of the special linear group over polynomial rings, *Math. USSR-Izv.* **11**, No. 2 (1977).
12. R. G. SWAN, The number of generators of a module, *Math. Z.* **102** (1967), 318–322.
13. R. G. SWAN, Vector bundles and projective modules, *Trans. Amer. Math. Soc.* **105** (1962), 264–277.
14. W. V. VASCONCELOS, Lecture notes on the conormal bundle of an ideal, in "Atlas V Escola de Algebra," IMPA, Rio de Janeiro, 1979.